The Adaptive Time-Frequency Distribution Using the Fractional Fourier Transform

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Résumé - Dans cet article nous allons présenter une nouvelle méthode temps-fréquence pour la reduction des terms d'interférence. On va considérer le cas des signaux multi-composants, pour lequel l'élimination des termes d'interférence est difficile à résoudre. Pour extraire les composants utiles, nous proposons une technique basée sur le concept de décomposition atomique. Nous définissons une structure algorithmique pyramidale, en envisageant deux buts: l'amélioration des performances du processus d'estimation et la réduction de la complexité. On met en évidence les avantages de cette méthode, en ce qui concerne l'estimation du taux de modulation linéaire et la complexité.

Abstract - *In this paper a new time-frequency interference terms cancellation method will be presented. We will consider this problem in the multi-structure signal case, where this problem is often difficult to solve due to interference geometry, which can be superposed on the signal component. We propose a technique based on the atomic decomposition of the signal, in order to extract its components. A pyramidal algorithm structure is defined, in order to improve the parameter estimation performances and to reduce its complexity. So, we will show that the method offers some benefits, such as the more exact chirp rate estimation in the noisy environment. Beside the complexity improvement is obtained.*

I. Introduction

The most important interest for time-frequency representations (TFRs) is the efficient analysis procedure which can be done in the non-stationary environment. One primary motivation for these different schemes is to improve the joint time-frequency resolution with the least amount of cross-term interferences. Even if the Wigner-Ville Distribution (WVD) possesses the best time-frequency resolution, this transform has the problem of cross-term interferences, which greatly limits its applications. Naturally, many adaptive time-frequency schemes have been developed; one of this approach, introduced by Baraniuk [3], uses an adaptive gaussian kernel, designed in the ambiguity plane, which is the best matched on the time-frequency structure. In this case, a trade-off between the interference level and the signal component will limit the application of this approach.

An alternative approach is the adaptive time-frequency signal representation using a basis function dictionary, which is well matched on the time-frequency signal atoms. Actually we use a four parameter dictionary, which has the chirplet as the elementary functions. The chirplet is a generalization of the Gabor logon by the introduction of the frequency rate and scale parameters. In order to obtain this analyzing function two approaches can be developed.

The first solution is the application of time-shift, frequency-shift, scaling and chirp multiplication operators on the gaussian atom; the optimal choice of those parameters can be done by a likelihood maximization (LM) procedure. The operation of this method which provides good performances, supposes the knowledge of the atom number that composes the signal (research space dimension). The second solution consists in fractional Fourier transform (FRFT), which represents a generalization of the classical Fourier Transform.

This transform provides a measure for the angular distribution of energy in the time-frequency plane. So, we can success the rotation in time-frequency plane using the operator issued from this transform. Consequently, the four-parameter time-frequency atom will be obtained by the scaling, rotation, time-shift and frequency-shift operators, applied to the unit gaussian function.

In this paper we will firstly present the adaptive time-frequency dictionary construction procedure, using the FRFT. We will highlight their chirp rate estimation capability and a objective comparison will be done. So on, we propose a new algorithm to design the signal representation. This approach is based on the pyramidal algorithm, similar with the wavelet packet decomposition algorithm. We will use the FRFT to design the filter bank which will be used for the decomposition and we will make some considerations over this procedure in order to achieve an orthogonal transform.

II. The decomposition basis selection using the FRFT

2.1. FRFT definition and its time-frequency properties

The fractional Fourier transform is the generalization of the classical Fourier transform. It provides a measure for the angular distribution of energy in the time-frequency plane. The FRFT depends on a parameter a and can be interpreted as a rotation by an angle a in the time-frequency plane [1]. As shown in figure 1, the FRFT can be interpreted as the counterclockwise rotation of the signal representation by an arbitrary angle a.

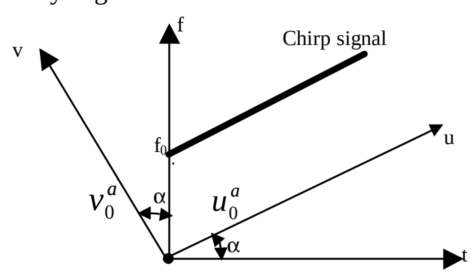


FIG. 1 : The fractional Fourier transform definition:

The FRFT of a signal *f(t)* is presented as :

$$FRFT_a\{f\} \overset{\Delta}{=} \sqrt{\frac{1-j\cot a}{2p}}\, e^{j\frac{\cot a}{2}x^2} \int_{-\infty}^{\infty} f(t)\, e^{j\frac{\cot a}{2}t^2} e^{-j\csc a\cdot tx} dt \quad (1)$$

$$FRFT_a\{f\} = (\Gamma_{-a} f)(x) = f_{-a}(x)$$

where $p = \dfrac{2p}{a}$ is the order of the FRFT. The relationship between the time-frequency origin (in standard time-frequency plane - a=0) and the fractional ones (in the origin centered chirp signal case, with a - chirp rate), may be expressed as follows (see figure 1):

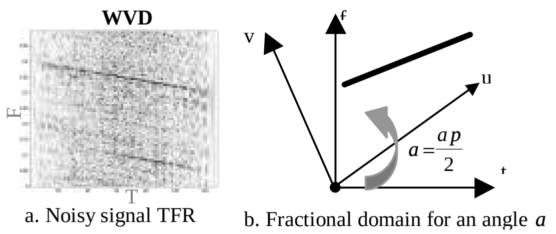
$$\begin{aligned} v_0^a &= f_0 \cos a \\ u_0^a &= f_0 \sin a \end{aligned} \quad (2)$$

Main time-frequency properties

The most important FRFT property is the representation of a chirp signal in the fractional domain. So, a chirp signal in the standard time-frequency plane will be represented by a tone. This property allows the possibility of chirp rate and the start frequency estimation, using the classical spectral estimation methods designed for sinusoidal signals. In the next figure we show the chirp parameter estimation procedure and the corresponding results obtained for a noisy chirp signal with the following parameters: *chirp_rate=-0.2, start_frequency=0.4, SNR=-1 dB.*

The procedure consists in the signal FRFT analyzing for some values of the angle a). We will retain the value which leads to a single and pronounced maximum; for this value we will evaluate the start frequency of the modulation. The temporal signal origin can be deduced from the coordinates origin taken into account for FRFT evaluation. The signal length can be estimated using a likelihood ratio maximization procedure, for example.

This procedure provides good results since the estimated parameter values are relatively closed for the real ones, presented above.



a. Noisy signal TFR   b. Fractional domain for an angle a

$$\begin{aligned} Chirp\ Rate &= tg\left[(1+a)\frac{p}{2}\right] \\ Start\_Freq: &= f_0 = \frac{u_0^a}{\cos\left(\frac{ap}{2}\right)} \end{aligned} \quad (3)$$

FIG. 2 : Chirp parameter estimation using FRFT

2.2. Linear frequency modulation of the wavelet functions

In [2], Baraniuk has introduced the scale-shear fan bases concept, defined in the Fourier domain of a wavelet basis element:

$$B^{wavelet}_{m,n}(f) = \left(Fb^{wavelet}_{m,n}\right)(f) = a_0^{M/2} G_{wavelet}\left(a_0^m f\right) e^{-j2pnt_0 a_0^m f} \quad (4)$$

F stands for the Fourier transform operator and $G_{wavelet}$ represents the Fourier transform of $g_{wavelet}$.

The scale-shear fan bases are constructed simply by replacing the linear f term in the exponential of (4) with another power of *f:*

$$B^{fan}_{m,n}(f) = r_0^{M/2} G_{fan}\left(r_0^m f\right) e^{-j2pnp_0\left|r_0^m f\right|^c sign(f)}, c \in \Re, c \neq 0 \quad (5)$$

Taking the inverse Fourier transform of $B^{fan}_{m,n}$ yields the proposed fan basis element of order c:

$$b^{fan}_{m,n}(t) = \left(F^{-1}B^{fan}_{m,n}\right)(t) = r_0^{-M/2}\left(C^c_{np_0} g_{fan}\right)\left(r_0^{-m}t\right) \quad (6)$$

The operator C_k^c represents a convolution with a hyper-chirp function of order c and chirp rate k, that is

$$\left(C_k^c g\right)(t) = \left(g * h_k^c\right)(t) \quad (7)$$

with

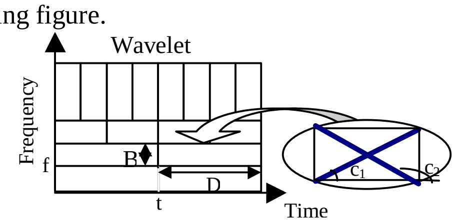
$$h_k^c(t) = \left(F^{-1} e^{-j2pk|f|^c sign(f)}\right)(t) = \int e^{-j2pk|f|^c sign(f)} e^{j2pft} df \quad (8)$$

Equation (6) indicates that the building blocks of a fan basis are obtained by convolving a fixed function g_{fan} with a chirp function of rate np_0 and then scaling the result. The chirp convolution causes the basis elements to *shear* in the time-frequency plane. Different values of the order parameter c correspond to different types of chirps and, hence, produce completely different time-frequency plane tillings [1]. In this work, we will employ only the linear chirps. Consequently, the algorithm to obtain the linear modulated wavelet function consists in the convolution between a wavelet basis function (at scale l, sub-band n and shift k) and a chirp function, generated by (8).

- **Chirp rate selection**

In order to apply this algorithm we need to establish a procedure to choose the optimal chirp rate, in the sense of best matching on the signal structure. We propose an estimation procedure based on the signal sub-space projection on the FRFT plane.

The main idea involved for this procedure is to compare the signal sub-space approximation with a reference chirp (with a positive and negative chirp rate) bounded with the wavelet packet tilling. The parameters which characterize these chirp functions are presented in the following figure.



Time-Frequency parameters:

Arbitrary WP Tilling:
- Level: l;
- Block index: n;
- Signal length:L.

- Start time: $t_0 = \dfrac{n}{2^l}L$
- Start frequency : $f_0 = \dfrac{1}{2^{L-l+1}}$
- Timewidth : $D = \dfrac{L}{2^l}$
- Bandwidth : $B = \dfrac{L}{2^{L-l+1}}$

FIG. 3 : Reference chirp definition

Using the notations introduced in figure 3, we can define the chirp rates of the up- and down- chirps :

$$\text{Upchirp (solid line):} \quad c_1 = \arctan\left(\frac{B}{D}\right)$$
$$\text{Dowwn-chirp (dashed line):} \quad c_2 = \arctan\left(-\frac{B}{D}\right) \tag{9}$$

In the orthogonal case the signal wavelet packet sub-space may be computed as follows:

$$s_{\ln}(t) = \sum_{k=1}^{L/2^l} C_{l,n,k} \psi_{l,n,k} \tag{10}$$

where $(C_{l,n,k})$ is the set of wavelet packet coefficients and the $(\psi_{l,n,k})$ are the corresponding wavelet functions. In order to approximate the s_{ln} by the chirp functions, we will represent this one in the FRFT plane, according to the considered WP sub-space. This approximation stage will be done in two phases.

Firstly, we evaluate the nature of the chirp signal which may approximate our signal sub-space. This can be done by the signal FRFT under two directions, corresponding to a positive chirp rate and a negative chirp rate, respectively.

Using the wavelet packet coefficients from the best basis (we suppose as best basis selection procedure the Shannon's entropy minimization, [3]) and the corresponding wavelet functions, we compute the signal wavelet packet subspace representation. The resulting shape will be transformed in the fractional coordinates. According to the FRFT property, the reference chirps will be interpreted, in the corresponding fractional domain, as a sinusoid with the frequency given by (3). After the evaluation of the fractional spectra WV distribution, we will compute the frequency marginal. Theoretically, for the reference chirp signal, we must obtain a Dirac pulse.

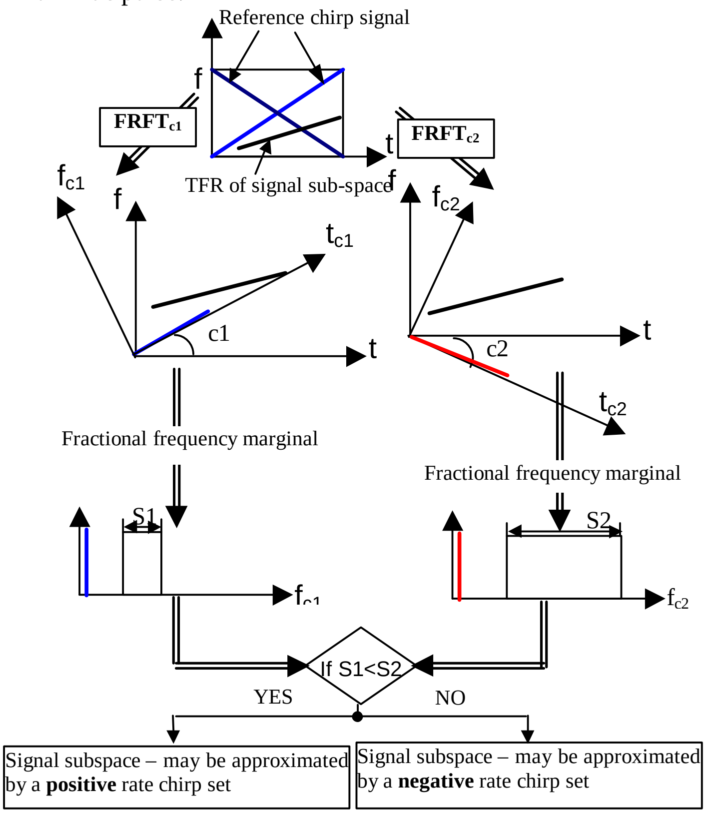


FIG. 4 : Chirp estimation principle from WP sub-space, suing FRFT

Furthermore, in the both FRFT domain, we compute the WVDs of the $s_{\ln}^{c1}$ and $s_{\ln}^{c2}$. After the fractional frequency marginal computing, the frequency spreading S1 and S2 will be evaluated. If the S1<S2 (the signal subspace is more concentred in the TF domain corresponding to up-chirp) we can conclude that this subspace may be approximated by a up-chirp. Otherwise, the situation becomes similar and the subspace approximation space will be a negative rate chirp. In figure 4, we have considered a simplicity example when the subspace TFR has a linear shape.

Using this procedure, we applied a dichotomy procedure to reduce a research space to only the positive (or negative) rate chirps.

In the second stage we try to estimate the linear modulation parameters which will be applied to corresponding wavelet functions. That is, using the corresponding coefficients of this subspace, we approximate the subspace fractional frequency marginal with a gaussian distribution, in order to extract the energetic center coordinates. In this sense, we use the following procedure:

$$\left.\begin{array}{l} SS = \frac{1}{M}\sum_{t_{c1}} WVD_{s_{\ln}^{c1}}(t \\ M = Max\left(WVD_{s_{\ln}^{c1}}\right) \\ f = 0 : F_c - 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f_m = \frac{\sum(f \cdot SS)}{Size(SS)} \\ S = 2\sqrt{p\frac{\sum((f - f_m}{Size(S}} \end{array}\right.$$

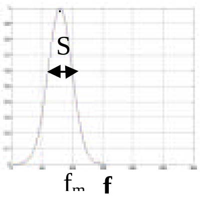


FIG. 5 : Fractional frequency marginal approximation by a Gaussian distribution

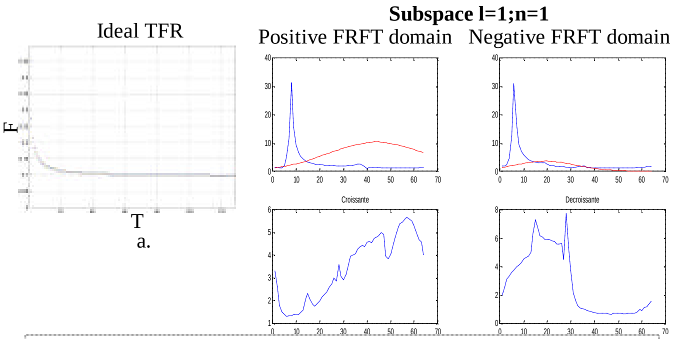
Using the f_m computed like that we will apply the procedure described in figure 2: we will turn the fractional coordinates and we will retain the FRFT orders for which the time-frequency concentrations become maximal. These values will be used to linearly modulate the wavelet functions corresponding to the central points of the Gaussian distribution. The number of functions that will be modulated depends on the spread parameter S. For example, if S is small, we will modulate only the function corresponding to the central point.

III. Orhotgonal Fractional Pyramidal Decomposition Algorithm (FPDA)

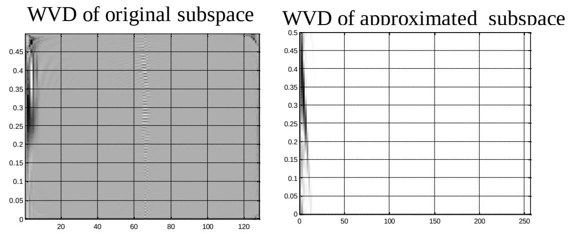
In many practical applications, the orthogonality is suitable from different points of view. First, there are some fast algorithms (such Mallat's pyramidal algorithm), which are well adapted for practical implementation. Secondly, the orthogonality of a basis ensures the transform coefficients decorrelation, which leads to specific applications, like signal denoising and compression. Beside, the basis orthogonality guarantees the representation unicity.

Unfortunately, an orthogonal basis is not always able to extract the signal features. In this section, we intend to use the algorithm previously presented in order to enhance the time-frequency image quality, in the orthogonal context.

This algorithm, called FPAD, due to the FRFT involving, begins by the signal WPD and the best basis selection, using the Shannon's entropy as cost function. After that, we will generate the corresponding signal subspaces. For each of them, we will apply the FRFT for both possible chirp rate, in order to evaluate the nature of chirp signal which can approximate a given subspace. After that, we will effectively search the optimal chirp rates and the most significant wavelet functions which will be used for the signal subspace approximation. The resulting function will be weighted with the best basis coefficients and we will compute the WVD of each retained basis function, having as objective the signal feature extraction, for the classification task. In order to highlight the stages of this algorithm, we intend to present the obtained results (for l=1;n=2 subspace) after each operation, using as test signal a hyperbolic chirp modulation, with an ideal TFR illustrated in the figure 6.a.



- Decision phase: approx. Shape - negative rate chirp
- Estimation phase: The wavelet functions $\psi_{1,1,2}$ will be modulated with c2=-0.82 and $\psi_{1,1,3}$ will be modulated with c2=-0.8



$$TFR_{subspace} = C_{1,1,2}^{2} \cdot WVD(\psi_{1,1,2}) + C_{1,1,3}^{2} \cdot WVD(\psi_{1,1,3})$$

b.

FIG. 6 : FPAD for subspace l=1;n=1

In the figure 6.b we present the case of a subspace extracted from the best basis. This case corresponds to the first part of hyperbolic chirp modulation, we apply the FRFTs for the orders defined in (9), and we compute the fractional frequency marginal. The spreading values show that this subspace can be approximated by a negative rate chirp set. After the estimation phase (where we turn our TF axes with different angular values, in the negative sense) and using the most important coefficients (2 and 3) we obtain the optimal values of the chirp rate which will be used for the corresponding wavelet function modulation. We observe in the same figure that the issued TFR is more concentrated that the original one.

In the next table, we present the corresponding values obtained for the other best basis subspaces.

TAB. 1 : Chirp rate estimation for WP subspace approximations

Subspace	**Nature of approximation chirps**	**Chirp rates estimates**	**Modulated wavelet functions**
l=1; n=1	Negative	c = -0.82; c = -0.8	$\psi_{1,1,2}$ & $\psi_{1,1,3}$
l=4; n=3	Negative	c = -0.01	$\psi_{4,3,3}$
l=4; n=4	Negative	c = -0.008	$\psi_{4,4,14}$
l=4; n=5	Negative	c=-0.43	$\psi_{4,5,5}$

In fact, using these chirp rates, we adapt our time-frequency partition in order to match better the processed signal. This feature is illustrated in the next figure, where we picture the phase plane issued from the classical WVD and from the FPBA-PA.

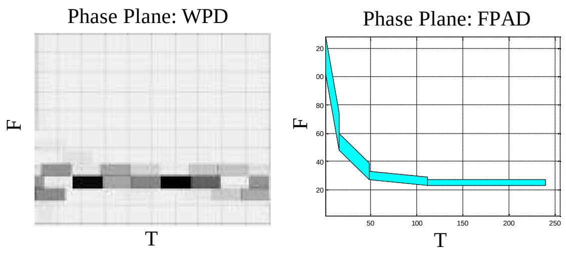


FIG. 7 : Phase planes issued by WPD and FPBA-PA

On this figure we observe that the FPAD partition is more adapted to model our test signal. Furthermore, the obtained tillings were used to search the optimal shapes which characterize the signal singularity. This property ensures the adaptability of the new time-frequency distribution on the time-frequency signal structure.

In figure 8 we plot the comparative results for the hyperbolic chirp modulation considered as the test signal.

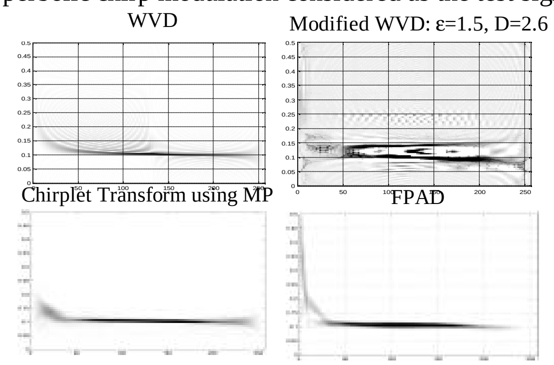


FIG. 8 : Comparative results for the hyperbolic chirp modulation

IV. Conclusion

Based on the results presented in figure 8, we can conclude that the FPAD provides a good time-frequency representation. Furthermore, we have eliminated the specific drawbacks of the MWVD (regarding the parameter set choice) or the chirplet transform (regarding the research space dimension setting).

In the classical WPD case, the use of the Mallat' pyramidal algorithm and the best basis procedure, based on the Shannon's entropy, guaranties the orthogonal behavior of this basis. In the FPBA-PA case, we have used the same method, but we have separately processed the significant wavelet function, in order to precisely extract the time-frequency signal features. For this purpose, we have modulated these functions using the chirp rates, extracted from the FRFT domain. In [2], it is shown that, if we use a orthogonal wavelet base, the modulation operator; described in the section 2.2., will conserve the orthogonal behavior for the new basis.

References

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